NPS-51Wu72031A

NAVAL POSTGRADUATE SCHOOL

Monterey, California

PHASE SPEED ERRORS WITH SECOND AND FOURTH ORDER SPACE DIFFERENCES WITH STAGGERED AND UNSTAGGERED GRIDS

by

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21 March 1972

Approved for public release; distribution unlimited

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ABSTRACT:

The linearized barotropic primitive equations are used to investigate the phase speed errors which arise from space truncation. Phase speed errors are compared for second and fourth order approximations to the space derivatives. These errors are also investigated on two staggered grids.

This task was supported by: Environmental Prediction Research Facility and Fleet Numerical Weather Central.

NPS-51Wu72031A
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1. Introduction

The objective of this report is to examine methods for improving the phase speeds of meteorological waves which are predicted by primitive equation models. In particular, the linearized phase speeds for second order and fourth order space difference approximations will be compared. When variables are computed at different grid points, computer time savings can be achieved. The phase speeds for two staggered grid schemes will be examined in an effort to find the scheme which will give the best phase prediction for the least computer time.

2. Phase Error Analysis

The analysis will be applied to the barotropic primitive equations; the results can then be easily extended to the baroclinic primitive equations. The equation of motion and continuity equation can be written:

$$\frac{\partial}{\partial t} \underset{\sim}{V} + \underset{\sim}{V} \cdot \nabla \underset{\sim}{V} + \nabla \phi + f\underset{\sim}{k} \times \underset{\sim}{V} = 0, \tag{1}$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underset{\sim}{V}) = 0 , \tag{2}$$

where $\phi = gH$; the quantity H is the height of the free surface.

These equations can be linearized by writing:

$$V = \left[U + u\,(x,t) \right] \underset{\sim}{i} + v\,(x,t)\, \underset{\sim}{j} , \tag{3}$$

$$\phi = \Phi - fUy + \varphi(x,t) ,$$

where U and Φ are constants and u, v and φ are very small. If we substitute (3) into (1) and (2) and drop products of small quantities, we obtain the following set of linearized equations:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} - fv = 0\ , \tag{4}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu = 0, \tag{5}$$

$$\frac{\partial \varphi}{\partial t} + U\frac{\partial \varphi}{\partial x} + \Phi\frac{\partial u}{\partial x} = 0. \tag{6}$$

We have neglected a term in the continuity equation which is proportional to $\partial \phi/\partial y$. The elimination of this term allows a simple separation of the modes, but it does not change their essential character.

If we eliminate between Eqs. (4) - (6) we obtain the following equation for φ :

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 + f^2 - \Phi\frac{\partial^2}{\partial x^2}\right]\varphi = 0. \tag{7}$$

The phase speeds of the waves can be obtained if we insert the wave form, $\varphi = Ae^{ik(x-ct)}$ into Eq. (7). The three solutions are given by

$$c = \begin{cases} U \\ U \pm \sqrt{\Phi + f^2/k^2} \end{cases} . \tag{8}$$

The first solution is the meteorological mode and the other two are inertial gravity waves.

Our first objective is to compare the phase speeds of the meteorological modes computed from prediction equations which have second and fourth order approximations to the space derivatives. This analysis can be conveniently carried out in a semi-discrete system where the spatial variation is discrete and the time variation is continuous. The phase speeds computed from the semi-discrete system will be very close to those computed from the fully discrete system. It can be shown that for the meteorological mode, the difference between the phase speeds in the discrete and the semi-discrete systems is proportional to $(U\Delta t/\Delta x)^2$. This ratio is very small because Δt must be small enough to keep the rapidly moving gravity waves stable. A stability analysis will be carried out later which will compare the values of Δt which are needed for stability for the second and fourth order space difference schemes.

If we write $x = j\Delta x$ and introduce second order differences into Eqs. (4) - (6), they become:

$$\frac{\partial u_j}{\partial t} + U\frac{(u_{j+1} - u_{j-1})}{2\Delta x} + \frac{(\varphi_{j+1} - \varphi_{j-1})}{2\Delta x} - fv_j = 0, \tag{9}$$

$$\frac{\partial v_j}{\partial t} + U\frac{(v_{j+1} - v_{j-1})}{2\Delta x} + fu_j = 0, \tag{10}$$

$$\frac{\partial \varphi_j}{\partial t} + U\frac{(\varphi_{j+1} - \varphi_{j-1})}{2\Delta x} + \Phi\frac{(u_{j+1} - u_{j-1})}{2\Delta x} = 0. \tag{11}$$

Each field is now expressed in terms of a wave of wave number k:

$$\begin{aligned} u_j &= \hat{u}(t)\, e^{ik\Delta xj}, \\ v_j &= \hat{v}(t)\, e^{ik\Delta xj}, \\ \varphi_j &= \hat{\varphi}(t)\, e^{ik\Delta xj}. \end{aligned} \tag{12}$$

If we insert these expressions into (9) - (11) and divide out the exponential, we obtain:

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{u} + i\alpha\hat{\varphi} - f\hat{v} = 0, \tag{13}$$

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{v} + f\hat{u} = 0, \tag{14}$$

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\hat{\varphi} + i\Phi\alpha\hat{u} = 0, \tag{15}$$

where $\alpha = (\sin k\Delta x)/\Delta x$. Eliminate between these equations to obtain the following equation for $\hat{\varphi}$:

$$\left(\frac{\partial}{\partial t} + i\alpha U\right)\left[\left(\frac{\partial}{\partial t} + i\alpha U\right)^2 + f^2 + \alpha^2 \Phi\right]\hat{\varphi} = 0. \tag{16}$$

The solution is of the form

$$\hat{u} = \hat{u}_0 e^{-ikct}$$

where

$$c = \begin{cases} \alpha U/k \\ \alpha U/k \pm \sqrt{f^2/k^2 + (\alpha^2/k^2)\,\Phi} \end{cases} \tag{17}$$

Since $\alpha \rightarrow k$ as $\Delta x \rightarrow 0$, these expressions converge to the exact expressions (8).

If we introduce fourth order space differencing into Eqs. (4) - (6), they become:

$$\frac{\partial u_j}{\partial t} + \frac{U}{12\Delta x}\left[8\,(u_{j+1}-u_{j-1}) - (u_{j+2}-u_{j-2})\right]$$

$$+ \frac{1}{12\Delta x}\left[8\,(\varphi_{j+1}-\varphi_{j-1}) - (\varphi_{j+2}-\varphi_{j-2})\right] - fv_j = 0, \tag{18}$$

$$\frac{\partial v_j}{\partial t} + \frac{U}{12\Delta x}\left[8\,(v_{j+1}-v_{j-1}) - (v_{j+2}-v_{j-2})\right] + fu_j = 0, \tag{19}$$

$$\frac{\partial \varphi_j}{\partial t} + \frac{U}{12\Delta x}\left[8\,(\varphi_{j+1}-\varphi_{j-1}) - (\varphi_{j+2}-\varphi_{j-2})\right]$$

$$+ \frac{\Phi}{12\Delta x}\left[8\,(u_{j+1}-u_{j-1}) - (u_{j+2}-u_{j-2})\right] = 0. \tag{20}$$

If we introduce the relations (12) into these equations, they will take the same form as equations (13) - (15) if we replace α by $\gamma = 4/3(\sin k\Delta x/\Delta x) - \frac{1}{6}\left[\sin (2k\Delta x)/\Delta x\right]$. In this case, the solutions for the phase speed become:

$$c = \begin{cases} (\gamma/k)\,U \\ \\ (\gamma/k)\,U \pm \sqrt{f^2/k^2 + (\gamma^2/k^2)\,\Phi} \end{cases} \tag{21}$$

These solutions also converge to the true phase speeds as $\Delta x \rightarrow 0$.

We are concerned with the phase speed of the meteorological mode whose exact value is U in this system. If U' is the phase speed of this mode in the semi-discrete system, we can write:

$$U' = U \sin k\Delta x / (k\Delta x) \tag{22}$$
(second order space difference),

$$U' = U \sin k\Delta x / (k\Delta x)\ \ (1 + 2/3 \sin^2 (k\Delta x/2)) \tag{23}$$
(fourth order space difference).

Fig. 1 shows the errors incurred for each scheme as a function of $k\Delta x$. This figure shows a considerable improvement for the fourth order scheme over the second order scheme except for the very short waves.

Let us now compare the linear computational stability criteria for the second and fourth order schemes. Since the criteria is mainly determined by the gravity waves, we will only consider the following simplified system:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi}{\partial x} = 0, \tag{24}$$

$$\frac{\partial \varphi}{\partial t} + \Phi \frac{\partial u}{\partial x} = 0. \tag{25}$$

Here we have set $f = U = 0$. If we write $t = n\Delta t$ and $x = j\Delta x$ and use second order differences, these equations become:

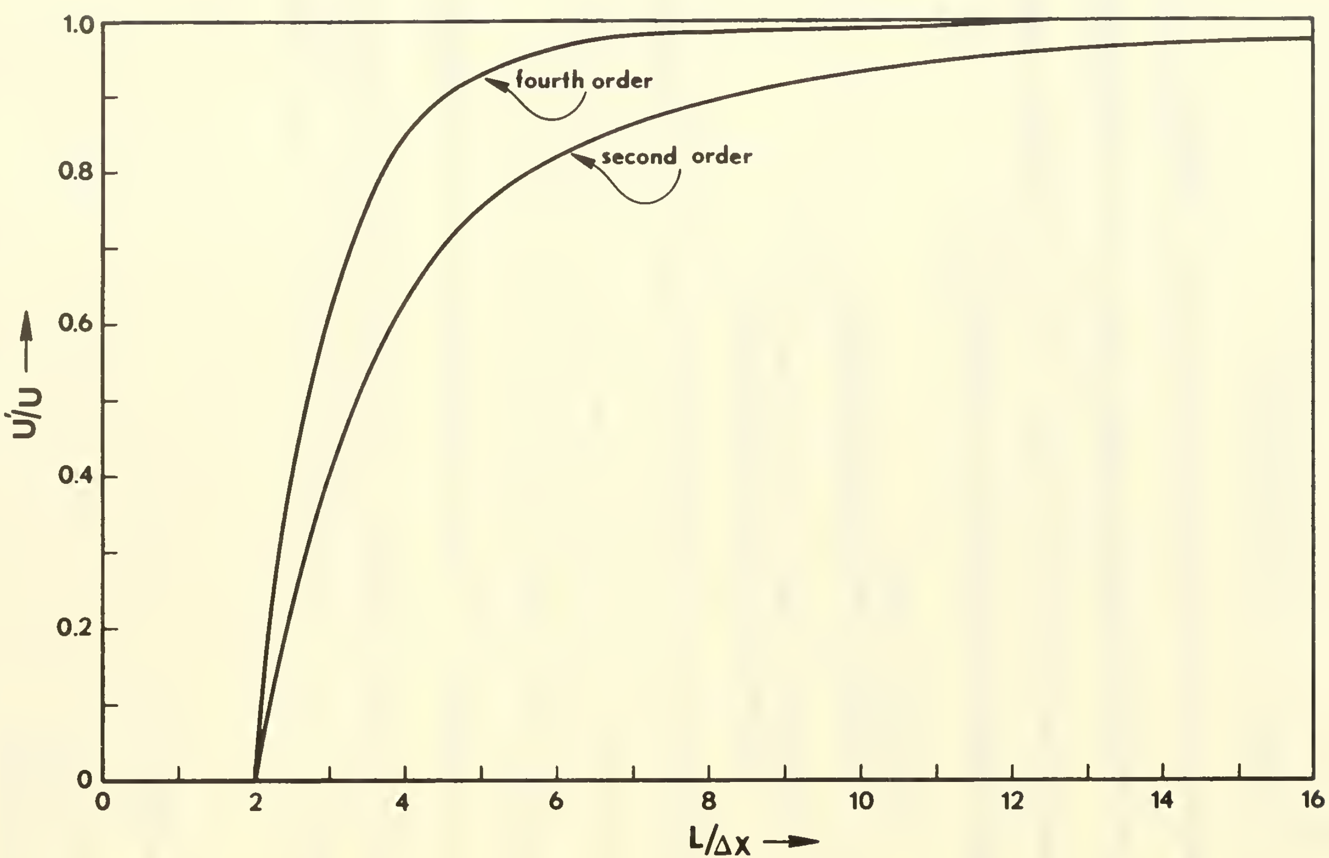
1.0
0.8
0.6
0.4
0.2
0
U'/U
0
2
4
6
8
10
12
14
16
L/Δx
fourth order
second order

Fig. 1

$$u_{jn+1} = u_{jn-1} - \frac{\Delta t}{\Delta x} (\varphi_{j+1n} - \varphi_{j-1n}) , \tag{26}$$

$$\varphi_{jn+1} = \varphi_{n-1} - \frac{\Delta t}{\Delta x} \Phi(u_{j+1n} - u_{j-1n}) . \tag{27}$$

We will employ the von Neumann method of stability analysis which is discussed by Haltiner (1971, p. 109). Introduce these Fourier relations:

$$u_{jn} = U_n e^{ik\Delta xj} , \varphi_{jn} = P_n e^{ik\Delta xj} . \tag{28}$$

If we substitute these expressions into (26) and (27), we will obtain:

$$U_{n+1} = U_{n-1} - \frac{2\Delta t}{\Delta x} i \sin k\Delta x \, P_n , \tag{29}$$

$$P_{n+1} = P_{n-1} - \frac{2\Phi\Delta t}{\Delta x} i \sin k\Delta x \, U_n . \tag{30}$$

Define the auxiliary variables:

$$A_n \equiv U_{n-1} , B_n \equiv P_{n-1} , \tag{31}$$

and

$$s \equiv (2\Delta t/\Delta x) \sin k\Delta x, r = 2 \Phi (\Delta t/\Delta x) \sin k\Delta x . \tag{32}$$

With these definitions, the difference equations can be written in two-level form as the following matrix equation:

$$\begin{pmatrix} U_{n+1} \\ A_{n+1} \\ P_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -is & 0 \\ 1 & 0 & 0 & 0 \\ -ir & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_n \\ A_n \\ P_n \\ B_n \end{pmatrix} \tag{33}$$

The 4 x 4 matrix in this equation is called the amplification matrix. The eigenvalues of the amplification matrix satisfy the following equation:

$$\lambda^4 - 2\,(1 - sr/2)\,\lambda^2 + 1 = 0. \tag{34}$$

This equation can be solved for λ^2 :

$$\lambda^2 = 1 - sr/2 \pm \sqrt{(1 - sr/2)^2 - 1}\,. \tag{35}$$

When $4 \geq sr \geq 0$, we find that $|\lambda^2| = 1$. If $sr > 4$, it is clear that $|\lambda^2| > 1$ for one of the roots. The difference equations will satisfy the von Neumann necessary condition for stability if we can choose Δt and Δx so that $4 \geq sr \geq 0$ for all k. From (31) the product sr is

$$sr = 4\,\Phi \left(\frac{\Delta t}{\Delta x} \right)^2 \sin^2 k\Delta x \quad . \tag{36}$$

It is clear from this expression that the second order scheme will be stable if

$$\Delta t \leq \Delta x / \Phi^{\frac{1}{2}}\ . \tag{37}$$

The difference forms of (24) and (25) with second order time differences and fourth order space differences are:

$$u_{jn+1} = u_{jn-1} - \frac{1}{6}\,\frac{\Delta t}{\Delta x} \left[8(\varphi_{j+1n} - \varphi_{j-1n}) - (\varphi_{j+2n} - \varphi_{j-2n}) \right], \tag{38}$$

$$\varphi_{jn+1} = \varphi_{jn-1} - \frac{1}{6}\,\frac{\Delta t \Phi}{\Delta x} \left[8(u_{j+1n} - u_{j-1n} - (u_{j+2n} - u_{j-2n}) \right]. \tag{39}$$

If the Fourier relations (28) are substituted into these equations, we obtain:

$$U_{n+1} = U_{n-1} - \frac{i\Delta t}{3\Delta x}\,(8 \sin k\Delta x - \sin 2k\Delta x)\, P_n\ , \tag{40}$$

$$P_{n+1} = P_{n-1} - \frac{i\Delta t\Phi}{3\Delta x}\,(8 \sin k\Delta x - \sin 2k\Delta x)\, U_n\ . \tag{41}$$

The matrix equation (33) is the same for this case if we define

$$\begin{aligned} s &= \Delta t/(3\Delta x) \left(8 \sin k\Delta x - \sin 2k\Delta x \right), \\ r &= \Phi \Delta t/(3\Delta x) \left(8 \sin k\Delta x - \sin 2k\Delta x \right). \end{aligned} \tag{42}$$

The solutions will be stable if $4 \geq sr \geq 0$ for all k. If we introduce the relations (42), this criteria becomes:

$$4 \geq \Phi \left(\frac{\Delta t}{\Delta x} \right)^2 4 \left(\frac{4}{3} \sin k\Delta x - \frac{1}{6} \sin 2k\Delta x \right)^2 \geq 0 \quad . \tag{43}$$

It can be shown that the maximum value of the quantity in the parenthesis is given by

$$\left(\frac{4}{3} \sin k\Delta x - \frac{1}{6} \sin 2k\Delta x \right)_{max} = 1.37 \quad . \tag{44}$$

The stability criteria for the fourth order space difference scheme now becomes

$$\Delta t \leq 0.73 \ \Delta x / \Phi^{\frac{1}{2}} \quad . \tag{45}$$

If we compare (45) and (37), we see that the fourth order scheme requires a time step which is smaller than the time step required for the second order scheme.

Let us now consider a mixture of second and fourth order differences which will allow a larger time step than the value given by (45). If the derivation of the phase speeds (17) and (21) is followed through, it can be seen that the phase speed of the meteorological wave is independent of the difference approximation which is used for the pressure force term. The pressure force differencing affects only the gravity waves in this model. Consider a difference set in which the advection terms and the divergence term in the continuity equation are computed with fourth order differences and the pressure force term is computed with second order differences. With this set the phase speed of the meteorological wave will still be given by (21). We now apply this difference set to the simplified Eqs. (24) and (25) in order to determine the critical value of the time step. The difference equations are given by (26) and (39) and the Fourier equations are given by (29) and (41). These equations can be written in the matrix form (33) if we define

$$s = (2\Delta t/\Delta x)\ \sin k\Delta x \ , \tag{46}$$

$$r = (\Phi/3)\ (\Delta t/\Delta x)\ (8 \sin k\Delta x - \sin 2k\Delta x)\ .$$

In this case, the stability criteria is

$$4 \geq 4\Phi(\Delta t/\Delta x)^2 \left[\sin k\Delta x\ (4/3 \sin k\Delta x - 1/6 \sin 2k\Delta x) \right] \geq 0 \ , \tag{47}$$

for all k. It can be shown that the quantity in the brackets is non-negative and its maximum value is given by

$$\left[\sin k\Delta x\ (4/3 \sin k\Delta x - \frac{1}{6} \sin 2k\Delta x) \right]_{max} = 1.35 \ . \tag{48}$$

The stability criteria now becomes

$$\Delta t \leq 0.86\ \Delta x / \Phi^{\frac{1}{2}} \ . \tag{49}$$

This mixture of second and fourth order time differences allows a longer time step while giving fourth order accuracy in the phase speed of the meteorological wave.

3. Staggered Grid Schemes

We will now examine the phase speed errors in two staggered grid schemes. The first, which is shown in Fig. 2, is used in the Arakawa-Mintz General Circulation Model (see Langlois and Kwok, 1969).

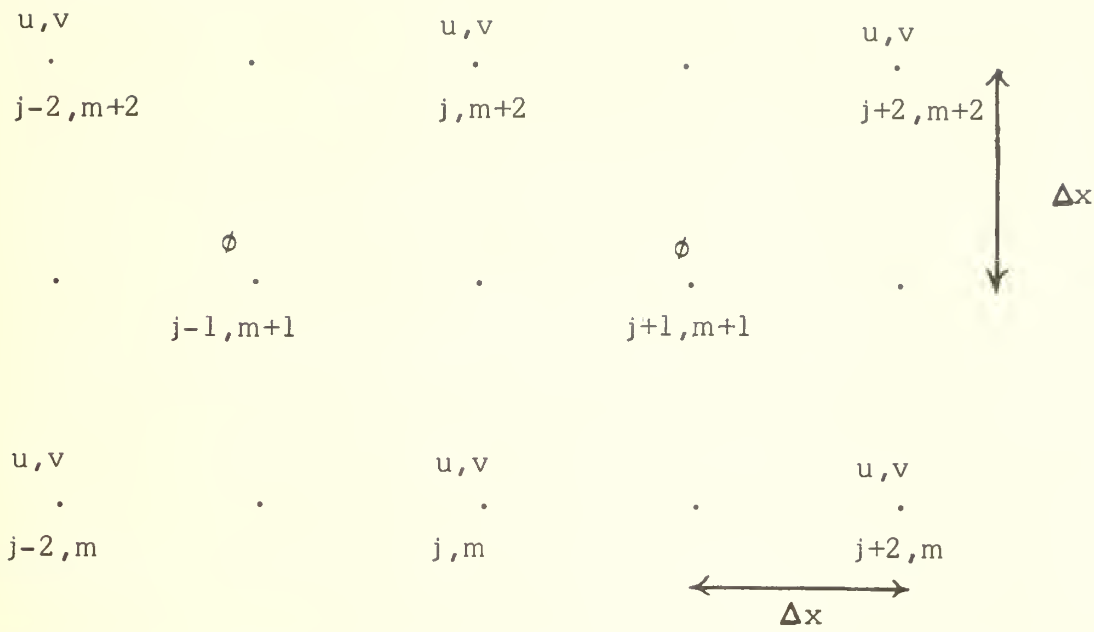

Fig. 2

If we compare numerical predictions on this grid to predictions on a similar grid where all variables are carried at every grid point, we find a reduction in computer time and memory size of 4 to 1. Although some averaging is required, the pressure force can be computed accurately since the differences are taken over a distance of $2\Delta x$. The advection terms are poorly represented, however, since the differences there are taken over a distance of $4\,\Delta x$. This gives a large error in the phase speed of the meteorological wave. The phase error for the meteorological wave of wavelength L on this grid is equal to the phase error of a wave of wavelength L/2 on the grid where all variables are carried at each point. Thus, for a wavelength of $6\,\Delta x$, the phase error ratio $U'/U = 0.41$ with this staggered grid while the value for the unstaggered scheme is $U'/U = 0.83$. If fourth order differences are used with the staggered grid, the phase error ratio is improved to $U'/U = 0.62$. This is still poorer than the value for the unstaggered grid. If we take into account the reduction in time step required by (45) or (49) and the increased computer time required to compute the more complicated advection terms, the staggered scheme with fourth order difference will require less computer time than the unstaggered scheme. This staggered grid scheme is very useful for general circulation experiments or for ocean current prediction (see Haney (1971), but it is less useful for operational short range prediction where the minimization of phase errors is very important.

Let us now consider the staggered grid array which is shown in Fig. 3 which was examined by Winninghoff (1968).

u,v · $j-1,m+1$	ϕ · $j,m+1$	u,v · $j+1,m+1$	
			Δx
ϕ · $j-1,m$	u,v · j,m	ϕ · $j+1,m$	
u,v · $j-1,m-1$	ϕ · $j,m-1$	u,v · $j+1,m-1$	
	Δx		

Fig. 3

This arrangement of grid points becomes the same as the arrangement shown in Fig. 2 if theaxesare rotated 45° and the grid size is increased. If we compare numerical predictions on this grid to predictions on a similar grid which is unstaggered, we find a reduction in computer time and memory size of 2 to 1. If a meteorological wave is propagating along either the x- or the y- axis on this grid, the phase error will be the same as on the unstaggered grid since the increments are taken over a distance of $2\,\Delta x$. This degree of accuracy requires the specific difference scheme which is presented later in this section. However, if the wave is propagating at an angle of 45° to one of the axes, the phase error will be larger since the differences are taken over a distance of $2\sqrt{2}\Delta x$. On the unstaggered grid, the errors are smaller since the differences are taken over a distance of $\sqrt{2}\,\Delta x$. Thus, for the 45° angle, the phase error on this staggered grid for a wave of wavelength L is equal to the phase error of a wave with wavelength $\frac{L}{2}$ on the unstaggered grid.

If we now introduce fourth order differences in the continuity equation and in the momentum advection terms, we will find an improvement in the phase speed of waves propagating along the axes as indicated in Fig. 1. The propagation in the 45° directions will be improved by a similar amount, but it will still contain more error than with the unstaggered scheme. Even with fourth order space differences, this scheme will require less computer time than for the unstaggered grid scheme. The phase propagation along the axes will be significantly better than for the second order unstaggered scheme, while the propagation in the 45° directions will be a little worse. Thus, this staggered scheme with fourth order space differences should give forecasts which are on the average better and require less computer time than the second order unstaggered scheme. Also, as pointed out by Winninghoff (1968), this staggered scheme gives a better description of the geostrophic adjustment process than the unstaggered scheme.

We will now present a set of difference equations for this staggered set of grid points. Equations (1) and (2) can be written in component form and combined to give the following three equations:

$$\frac{\partial}{\partial t}(u\phi) + \frac{\partial}{\partial x}(uu\phi) + \frac{\partial}{\partial y}(vu\phi) + \phi\frac{\partial \phi}{\partial x} - fv\phi = 0 \quad , \tag{50}$$

$$\frac{\partial}{\partial t}(v\phi) + \frac{\partial}{\partial x}(uv\phi) + \frac{\partial}{\partial y}(vv\phi) + \phi\frac{\partial \phi}{\partial y} + fu\phi = 0 \quad , \tag{51}$$

$$\partial \phi / \partial t + \frac{\partial}{\partial x}(u\phi) + \frac{\partial}{\partial y}(v\phi) = 0 \quad . \tag{52}$$

For simplicity, the time variation will be left continuous. If we are to take full advantage of the placement of points on this grid, it is necessary to compute the flux divergences on a coordinate system which has been rotated 45°. Define the new coordinates as follows:

$$\xi = \frac{1}{\sqrt{2}}(x + y) \quad , \tag{53}$$

$$\eta = \frac{-1}{\sqrt{2}}(x - y) \quad ,$$

and the corresponding velocity components are

$$\dot{\xi} = \frac{1}{\sqrt{2}}(u + v) \quad , \tag{54}$$

$$\dot{\eta} = \frac{1}{\sqrt{2}}(-u+v) \quad .$$

The flux divergence of the quantity s may be written

$$\frac{\partial}{\partial x}(u\phi s) + \frac{\partial}{\partial y}(v\phi s) = \frac{\partial(\dot{\xi}\phi s)}{\partial \xi} + \frac{\partial(\dot{\eta}\phi s)}{\partial \eta} \quad , \tag{55}$$

where s = u or v in equations (50) and (51).

Fig. 4 contains the new coordinates superimposed upon the staggered grid.

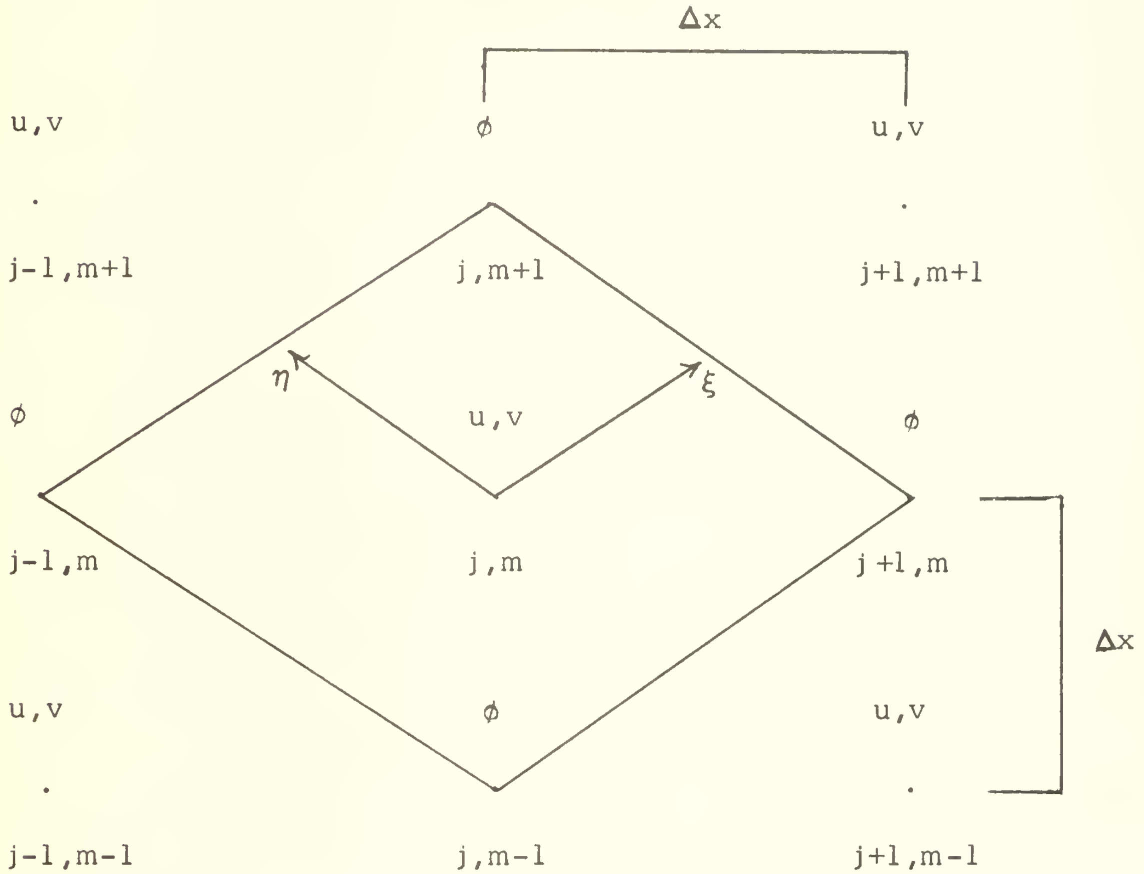

Fig. 4

The flux terms (55) are computed in terms of fluxes across the sides of the diamond shown in Fig. 4. In the finite difference formulation we require values for ϕ at some u, v points. Thus, we define

$$\bar{\phi}_{jm} = \tfrac{1}{4}\left(\phi_{j+1m} + \phi_{j-1m} + \phi_{jm+1} + \phi_{jm-1}\right) \qquad (56)$$

The finite difference forms of (50), (51), and (52) with second order differences are:

$$\frac{\partial}{\partial t}(u\bar{\phi})_{jm} + \frac{1}{8\Delta x}\Big\{\Big[u_{j+1\,m+1}+u_{jm}\Big]\Big[(u\bar{\phi})_{j+1m+1}+(u\bar{\phi})_{jm}+(v\bar{\phi})_{j+1m+1}+(v\bar{\phi})_{jm}\Big]$$

$$-\Big[u_{j-1\,m-1}+u_{jm}\Big]\Big[(u\bar{\phi})_{jm}+(u\bar{\phi})_{j-1m-1}+(v\bar{\phi})_{jm}+(v\bar{\phi})_{j-1m-1}\Big]$$

$$+\Big[u_{j-1m+1}+u_{jm}\Big]\Big[-(u\bar{\phi})_{j-1m+1}-(u\bar{\phi})_{jm}+(v\bar{\phi})_{j-1m+1}+(v\bar{\phi})_{jm}\Big]$$

$$-\Big[u_{jm}+u_{j+1\,m-1}\Big]\Big[-(u\bar{\phi})_{jm}-(u\bar{\phi})_{j+1m-1}+(v\bar{\phi})_{jm}+(v\bar{\phi})_{j+1m-1}\Big]\Big\}$$

$$+\bar{\phi}_{jm}(\phi_{j+1m}-\phi_{j-1m})/2\Delta x - f(v\bar{\phi})_{jm} = 0, \tag{57}$$

$$\frac{\partial}{\partial t}(v\bar{\phi})_{jm} + \frac{1}{8\Delta x}\Big\{\Big[v_{j+1\,m+1}+v_{jm}\Big]\Big[(u\bar{\phi})_{j+1m+1}+(u\bar{\phi})_{jm}+(v\bar{\phi})_{j+1m+1}+(v\bar{\phi})_{jm}\Big]$$

$$-\Big[v_{j-1m-1}+v_{jm}\Big]\Big[(u\bar{\phi})_{jm}+(u\bar{\phi})_{j-1m-1}+(v\bar{\phi})_{jm}+(v\bar{\phi})_{j-1m-1}\Big]$$

$$+\Big[v_{j-1m+1}+v_{jm}\Big]\Big[-(u\bar{\phi})_{j-1m+1}-(u\bar{\phi})_{jm}+(v\bar{\phi})_{j-1m+1}+(v\bar{\phi})_{jm}\Big]$$

$$-\Big[v_{jm}+v_{j+1m-1}\Big]\Big[-(u\bar{\phi})_{jm}-(u\bar{\phi})_{j+1m-1}+(v\bar{\phi})_{jm}+(v\bar{\phi})_{j+1m-1}\Big]\Big\}$$

$$+\bar{\phi}_{jm}(\phi_{jm+1}-\phi_{jm-1})/2\Delta x + f(u\bar{\phi})_{jm} = 0, \tag{58}$$

$$\frac{\partial\bar{\phi}_{jm}}{\partial t} + \frac{1}{2\Delta x}\Big[(u\bar{\phi})_{j+1m}-(u\bar{\phi})_{j-1m}+(v\bar{\phi})_{jm+1}-(v\bar{\phi})_{jm-1}\Big] = 0. \tag{59}$$

Fourth order differences can be introduced into any term if the differences are replaced by the weighted average of the differences computed on this grid and the differences computed on a grid with a double Δx. The weights are 4/3 for the small grid and -1/3 for the large grid.

4. Conclusions

In this report we have shown that the use of fourth order differences in the momentum advection terms produces significant improvement in the phase speeds of short waves (see Fig. 1). The use of fourth order space differences, however, requires a time step which is 0.73 times the usual time step to maintain computational stability. If the pressure force terms are computed with the second order differences and the other terms with fourth order differences, the time step can be increased to 0.86 times usual time step. This change in the differencing for the pressure force term does not affect the phase speed of the meteorological wave.

Two staggered grid point arrangements were examined with respect to phase errors and computer time savings. The first scheme examined carries each variable at only one-quarter of the grid points with a corresponding savings in computer time. However, the advective phase errors for short waves are large in this system because the variables are differenced over twice as large a distance as with the unstaggered grid. Even the use of fourth order differencing will not bring the phase errors down to the value obtained from the unstaggered grid. The second scheme carries each variable at one-half of the grid points, which gives a savings of two in computer time. The phase errors on this grid are the same as on the unstaggered grid for waves moving along the coordinate axes. However, for waves of wavelength L moving at an angle of 45° to the axes, the phase error in this staggered grid is equal to the error with a wave of wavelength $\frac{L}{2}$ in the unstaggered grid. If fourth order differences are used, the overall phase propagation should be better than for the unstaggered grid. It is suggested that this scheme be tested with fourth order differences in all terms except the pressure gradient term. This term should be computed with second order differences to allow a longer time step. The finite difference equations are given for a scheme which computes flux divergence along lines which are at an angle of 45° to the coordinate axes. Winninghoff (1971) has tested a scheme on the sphere which has this grid point arrangement, but different advective approximations.

Acknowledgements

The author benefited from discussion with Drs. A. Arakawa, R. L. Elsberry, W. L. Gates, G. J. Haltiner, and R. L. Haney.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

Phase Speed Errors With Second And Fourth Order Space Differences With Staggered And Unstaggered Grids.

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report 1972

5. AUTHOR(S) *(First name, middle initial, last name)*

Roger T. Williams

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO OF REFS
	23	5

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) NPS-51Wu72031A
b. PROJECT NO. 55319 (EPRF)	
c. 55308 (FNWC)	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES This research was sponsored by Environmental Prediction Research Facility and Fleet Numerical Weather Central, Monterey, California.	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California

13. ABSTRACT

The linearized barotropic primitive equations are used to investigate the phase speed errors which arise from space truncation. Phase speed errors are compared for second and fourth order approximations to the space derivatives. These errors are also investigated on two staggered grids.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Numerical Weather Prediction						
Phase Speed Errors						
Staggered Grid Scheme						